A STOKES FLOW BOUNDARY INTEGRAL MEASUREMENT OF TUBULAR STRUCTURE CROSS SECTIONS IN TWO DIMENSIONS

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ABSTRACT

In this paper we will develop a method to determine cross sections of arbitrary two-dimensional tubular structures, which are allowed to branch, by means of a Stokes flow based boundary integral formulation. The measure for the cross sections for a point on the boundary of a given structure will be the path obtained by integrating perpendicularly to the flow lines from one side of the boundary to the other. Special emphasis will be put on the behavior at branching points, the behavior at vortices, and the necessary boundary conditions. The method can be extended to three dimensional problems.

1. INTRODUCTION

Measuring cross sections of structures in a consistent, meaningful way is important for many applications. In medical imaging, anatomical structures often times exhibit complicated, convoluted shapes. Manual thickness measurements based on image data (especially in three dimensions) then easily become error-prone. Jones et al. [1] use Laplace's equation to measure cortical thickness in three-dimensional images whose variations can be associated with many pathologies: e.g. Alzheimer's disease. Yezzi and Prince [2] improve the speed of Jones' computational method and eliminate the need for the explicit computation of trajectories. However, as in [1], the approach is confined to structures that can be described by two simply connected boundaries. Budding structures (see Section 5) pose serious problems for the aforementioned approaches. Note that, since the temperature at the two boundaries is fixed, no sensible thickness would be assigned to the cavity of the budding structure by a Laplace equation based method: a solution of Laplace's equation will not exhibit vortices.

This paper will deal with structures that cannot be described by two simply connected boundaries, e.g. structures that branch (e.g. blood vessels), and/or exhibit budding/constriction of boundary part(s). Instead of defining thickness by means of the solution of Laplace's equation we use the Stokes equation (describing fluid flow at low Reynolds numbers), with free slip boundary conditions. We assume that the object to be measured is given as a digital image. To solve the Stokes equation we use a boundary integral formulation, thus reducing the dimensionality of the problem by one dimension. There is no need to discretize the interior of the object. We restrict ourselves to the two dimensional problem (flow in a plane). However, the methodology developed can be extended to three dimensions (the boundary integral formulation for the Stokes equation is standard in three dimensions). Other fluid flow equations have recently been used in computer graphics and image processing, notably for image inpainting/reconstruction [3]. Examples for using the boundary element approach in computer vision can be found in [4].

Section 2 introduces the Stokes equations and presents the boundary integral formulation as given by Pozrikidis [5]. Section 3 deals with the discretization of the boundary integral equations, and introduces the formalism for solving the discretized equations. Section 4 describes the methodology for measuring cross sections once a flow field has been obtained. Examples are given in Section 5. The paper ends with a conclusion and suggestions for future work in Section 6.

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2. BOUNDARY INTEGRAL FORMULATION OF THE STOKES EQUATIONS

Pozrikidis [5] gives a good introduction to the boundary integral method for linearized viscous flow. Higdon [6] treats two-dimensional Stokes flow problems for arbitrarily shaped domains. Zeb et al. [7] present an implementation of Higdon's approach and extend it to handle pressure boundary conditions. This section aims at reviewing the basics for the formulation of the Stokes flow problem in terms of boundary integrals, following [5, 7].

2.1. The Stokes flow equations

Incompressible fluid flow problems where viscous forces dominate can be described by the divergence-free condition and the Stokes equation

$$\nabla \cdot \boldsymbol{u} = 0, \quad -\nabla P + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{b} = 0. \tag{1}$$

Here $\boldsymbol{u}$ is the fluid velocity, μ the viscosity of the fluid, P is the pressure and $\boldsymbol{b}$ is the body force. ∇ indicates the gradient, $\nabla\cdot$ represents the divergence operator, and ∇^2 the

Laplacian. In the sequel we incorporate the body force in the modified pressure $P^{\mathrm{MOD}} = P - \rho \boldsymbol{b} \cdot \boldsymbol{x}$. To avoid unnecessary heavy notation P is understood to mean P^{MOD} in what follows.

2.2. Boundary integral formulation

The incompressible solution of the Stokes equation

$$-\nabla P + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{g} \delta(\boldsymbol{x} - \boldsymbol{x}_0) = 0,$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ is a two-dimensional Dirac-delta-function centered at $\boldsymbol{x}_0$ and $\boldsymbol{g}$ is an arbitrary vector-valued constant, is given by[1]

$$u_i(\boldsymbol{x}) = \frac{1}{4\pi\mu} G_{ij}(\boldsymbol{x}, \boldsymbol{x}_0) g_j. \qquad (2)$$

(2) is the solution for the Stokes flow caused by a two-dimensional point force at $\boldsymbol{x}_0$. G_{ij} are the Green's functions, which describe the influence of a force at position $\boldsymbol{x}_0$ on the solution at position $\boldsymbol{x}$. Since the Stokes flow problem is linear, one can express the solution to an arbitrary force distribution by superposition of the respective Green's function solutions for each point force.

The stress associated with this flow may be written

$$\sigma_{ij}(\boldsymbol{x}) = \frac{1}{4\pi} T_{ijk}(\boldsymbol{x}, \boldsymbol{x}_0) g_k,$$

where the stress tensor T_{ijk} is

$$T_{ijk}(\boldsymbol{x}, \boldsymbol{x}_0) = - \delta_{ik} p_j(\boldsymbol{x}, \boldsymbol{x}_0) + \frac{\partial G_{ij}}{\partial x_k}(\boldsymbol{x}, \boldsymbol{x}_0) + \frac{\partial G_{kj}}{\partial x_i}(\boldsymbol{x}, \boldsymbol{x}_0).$$

The Green's functions G_{ij} (also called the two-dimensional Stokeslets) and the stress tensor T_{ijk} are given by [5]

$$G_{ij} = -\delta_{ij} \ln r + \frac{\hat{x}_i \hat{x}_j}{r^2}, \quad T_{ijk} = -4 \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{r^4}.$$

Here δ_{ik} is the Kronecker-delta-function, $r = \|\boldsymbol{x} - \boldsymbol{x}_0\|_2$, and $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$.

For a point $\boldsymbol{x}_0$ inside or on the boundary the solution to the Stokes equation (1) in terms of boundary integral equations is

$$\eta(\boldsymbol{x}_0) u_j(\boldsymbol{x}_0) = -\frac{1}{4\pi\mu} \int_{\partial D} f_i(\boldsymbol{x}) G_{ij}(\boldsymbol{x}, \boldsymbol{x}_0)\, \mathrm{d}l(\boldsymbol{x}) + \frac{1}{4\pi} \int_{\partial D} u_i(\boldsymbol{x}) T_{ijk}(\boldsymbol{x}, \boldsymbol{x}_0) n_k(\boldsymbol{x})\, \mathrm{d}l(\boldsymbol{x}), \qquad (3)$$

for the velocities, where $f_i(\boldsymbol{x}) = \sigma_{ij}(\boldsymbol{x}) n_j(\boldsymbol{x})$ is the stress force, and $\boldsymbol{n}(x)$ is the normal vector to the boundary ∂D at position $\boldsymbol{x}$, pointing towards the interior of the solution domain D. $\eta(\boldsymbol{x}_0)$ accounts for the dependency of the velocity integral on the location of $\boldsymbol{x}_0$: $\eta(\boldsymbol{x}_0) = 1$ if $\boldsymbol{x}_0 \in D$, $\eta(\boldsymbol{x}_0) = \frac{1}{2}$ if $\boldsymbol{x}_0 \in \partial D$.

[1] Note that we are looking at the two-dimensional Stokes flow problem. This is the solution of the two-dimensional equation, written with Einstein's summation convention (summing over multiply occurring indices). Solutions in three dimensions also exist, as treated in [5]. We denote vector components by subscripts, i.e. u_1 and u_2 for the components of $\boldsymbol{u}$.

3. BOUNDARY DISCRETIZATION

Following [7] we assume a piecewise linear parameterization of the boundary. The boundary ∂D is divided into n straight lines ∂D_k, $k = 1, 2, \ldots n$, where $f_i(\boldsymbol{x})$ and $u_i(\boldsymbol{x})$ are assumed constant along these elements. With these simplifications, the governing boundary integral equations (3) become

$$\eta(\boldsymbol{x}_0) \begin{pmatrix} u_1(\boldsymbol{x}_0) \\ u_2(\boldsymbol{x}_0) \end{pmatrix} = \frac{1}{4\pi} \sum_{k=1}^{n} n_k(\boldsymbol{x}_k) \begin{pmatrix} \tilde{T}_{11k}(\boldsymbol{x}_k, \boldsymbol{x}_0) & \tilde{T}_{21k}(\boldsymbol{x}_k, \boldsymbol{x}_0) \\ \tilde{T}_{12k}(\boldsymbol{x}_k, \boldsymbol{x}_0) & \tilde{T}_{22k}(\boldsymbol{x}_k, \boldsymbol{x}_0) \end{pmatrix} \begin{pmatrix} u_1(\boldsymbol{x}_k) \\ u_2(\boldsymbol{x}_k) \end{pmatrix} - \frac{1}{\mu} \begin{pmatrix} \tilde{G}_{11}(\boldsymbol{x}_k, \boldsymbol{x}_0) & \tilde{G}_{21}(\boldsymbol{x}_k, \boldsymbol{x}_0) \\ \tilde{G}_{12}(\boldsymbol{x}_k, \boldsymbol{x}_0) & \tilde{G}_{22}(\boldsymbol{x}_k, \boldsymbol{x}_0) \end{pmatrix} \begin{pmatrix} f_1(\boldsymbol{x}_k) \\ f_2(\boldsymbol{x}_k) \end{pmatrix} \qquad (4)$$

where the two dimensional Stokeslet integrals are given by

$$\begin{aligned} \tilde{G}_{11}(\boldsymbol{x}_k, \boldsymbol{x}_0) &= \int_{\partial D_k} - \ln r + \frac{\hat{x}_1 \hat{x}_1}{r^2}\, \mathrm{d}l \\ \tilde{G}_{12}(\boldsymbol{x}_k, \boldsymbol{x}_0) &= \int_{\partial D_k} \frac{\hat{x}_1 \hat{x}_2}{r^2}\, \mathrm{d}l = \tilde{G}_{21}(\boldsymbol{x}_k, \boldsymbol{x}_0) \\ \tilde{G}_{22}(\boldsymbol{x}_k, \boldsymbol{x}_0) &= \int_{\partial D_k} - \ln r + \frac{\hat{x}_2 \hat{x}_2}{r^2}\, \mathrm{d}l, \end{aligned} \qquad (5)$$

and the stress tensor integrals by

$$\begin{aligned} \tilde{T}_{11k}(\boldsymbol{x}_k, \boldsymbol{x}_0) n_k(\boldsymbol{x}_k) &= -4 \hat{x}_i n_i(\boldsymbol{x}_k) \int_{\partial D_k} \frac{\hat{x}_1 \hat{x}_1}{r^4}\, \mathrm{d}l \\ \tilde{T}_{12k}(\boldsymbol{x}_k, \boldsymbol{x}_0) n_k(\boldsymbol{x}_k) &= -4 \hat{x}_i n_i(\boldsymbol{x}_k) \int_{\partial D_k} \frac{\hat{x}_1 \hat{x}_2}{r^4}\, \mathrm{d}l = \tilde{T}_{21k} \\ \tilde{T}_{22k}(\boldsymbol{x}_k, \boldsymbol{x}_0) n_k(\boldsymbol{x}_k) &= -4 \hat{x}_i n_i(\boldsymbol{x}_k) \int_{\partial D_k} \frac{\hat{x}_2 \hat{x}_2}{r^4}\, \mathrm{d}l. \end{aligned} \qquad (6)$$

$\mathrm{d}l$ is the differential along the boundary element. Equations (5, 6) can be evaluated analytically; they describe the influence of a whole boundary segment on a point $\boldsymbol{x}_0$ located in or on the boundary of the domain D. The integrands of Equations (5, 6) exhibit singularities. These are removable (except for the case of points which lie exactly on the end point of a boundary element)[2].

To evaluate the discretized boundary integrals (4) we use the following two step approach (see [7]):

a) Evaluate the velocity boundary integrals for the n center points of the boundary elements, by specifying $2n$ boundary conditions. This is a linear equation system with $4n$ unknowns (f_1, f_2, u_1, u_2 for every boundary element) and $2n$ equations. Fast algorithms for solving such boundary integral equations exist, e.g. [8].

b) Solve for the velocity at arbitrary positions inside the domain D by using Equation (4).

[2] We omit the explicit analytical expressions for the integrals due to space constraints.

The boundary decomposes into the inlet, outlet and wall boundary parts: $\partial D = \partial D_i \cup \partial D_o \cup \partial D_w$. We prescribe the velocity profile $\overline{u}_1, \overline{u}_2$[3] at the inlet ($\partial D_i$) and assume stress free boundary conditions for the outlet(s) (i.e. $\overline{f}_1 = \overline{f}_2 = 0$ for $\boldsymbol{x} \in \partial D_o$). ∂D_w exhibits perfect slip without permeation: $\overline{u}_2 = 0, \overline{f}_1 = 0$. Note, that these boundary conditions directly translate to the three-dimensional case. While the Stokes equation itself is linear and reversible, we note that the generated flow fields will vary slightly under interchanging the identification of the inlet and outlet, because of the prescribed boundary conditions. These differences, however, will largely be restricted to the immediate vicinities of the inlet and outlet for reasonable prescribed inlet flows.

4. MEASUREMENT OF CROSS SECTIONS

To measure cross sections of a structure we propose to integrate perpendicular to the computed flow field. We compute

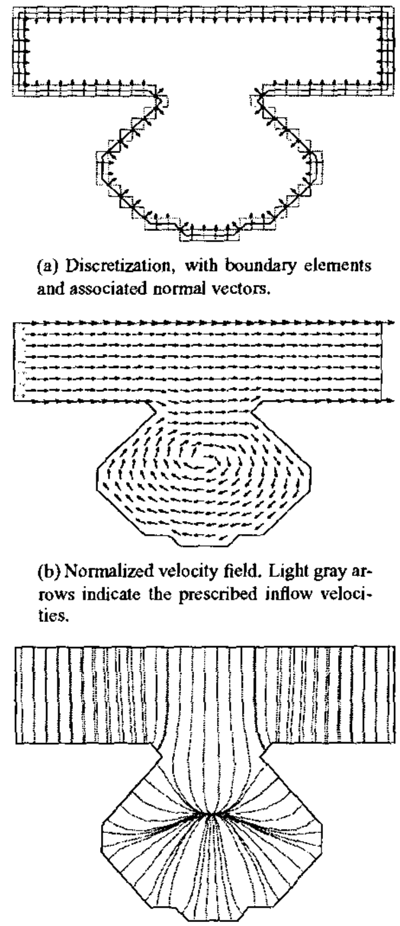
$$\boldsymbol{x}(l) = \int_0^l \boldsymbol{v} \, \mathrm{d}l, \qquad (7)$$

where $\boldsymbol{v} \in \mathbb{R}^2$ is a unit vector normal to the fluid speed $\boldsymbol{u}$ at all points, $\boldsymbol{x}(L) \in \partial D$ with $L \neq 0$, and $\boldsymbol{x}(0)$ is the boundary point for which the cross section is to be measured. L is then the path length from the initial to the final boundary point. We set the initial condition of the vector to the normal vector for the boundary element of the starting point, and disallow directional flips throughout the integration.

For the evaluation of (7) we use a modified version of the variable order Runge-Kutta method by Cash and Karp [9]. It is modified in the sense that a minimal step size h_{min} is introduced. Once the integration algorithm tries to use a step size $h \leq h_{min}$ we stop the integration and declare the current integrator state as the end point. In this way we stop the integration when either a boundary point or a stagnant point (where the step size will tend to zero) is reached. The combination of an integrator with adaptive step size and the boundary integral method thus yields a method that will naturally adapt to the problem at hand. If a lot is known about the shape of the boundary (i.e. we have a high resolution image) this information will be used, without requiring too many steps to integrate from an initial point on the boundary to the target point. For very thin structures (thickness in the order of a few pixels) the method will automatically result in subpixel accuracy.

5. EXAMPLES

This section presents examples for cross section measurements: a budding domain, and a spinous process of a vertebra. We set $\mu = 1$ and $h_{min} = 0.005$, and the relative error tolerance of the integrator to $\epsilon = 10^{-4}$. Figures 1(a) shows the discretized domain of the budding example, with its boundary elements and the associated normal vectors. Figures 1(b) and 2(a) show the normalized flow fields of the examples, and the computer tomography (CT) image of the spinous process[4]. Figures 1(c) and 2(b) show their respective measures for the cross sections. Of special interest is the behavior close to the branching point for the spinous process. While the Stokes flow based approach produced sensible results, a method based for example on finding the nearest point on the opposite boundary would lead to unreasonable results. The budding domain exhibits a vortex in its flow field (see Figure 1(b)). The path lines perpendicular to the flow field of the budding domain capture the rectangular part of the domain well. The cavity shows the end points of several path lines at the stagnant center of the vortex.



(a) Discretization, with boundary elements and associated normal vectors.

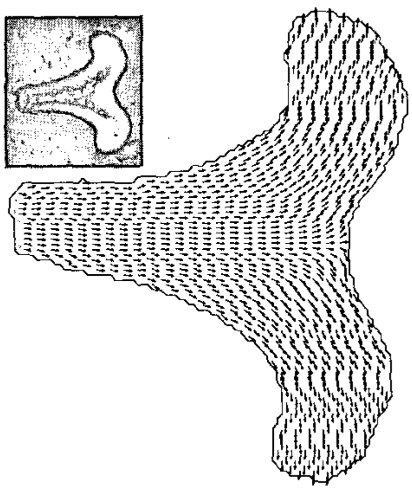
(b) Normalized velocity field. Light gray arrows indicate the prescribed inflow velocities.

(c) Cross sections.

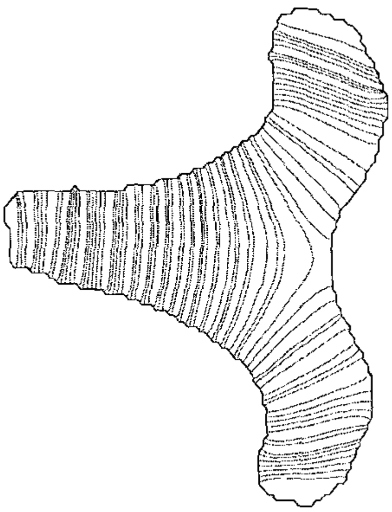
Fig. 1. The budding domain.

[3] Overlined variables denote variables given in a locally defined coordinate system of a boundary element, where the $\overline{x}_2$ direction coincides with the normal vector.

[4] The authors would like to thank Prof. Yamamoto for providing the image. The boundaries were extracted by means of an active contours based approach.



(a) Normalized velocity field and CT image. Light gray arrows indicate the prescribed inflow velocities.



(b) Cross sections.

Fig. 2. The spinous process.

One could for example connect these path lines by requiring minimal change of direction at the joining point. We see that through the presence of the vortex, a measure for the cross section of the budding part of the structure as well as of the rectangular part can be given. This would not have been possible with an approach based on Laplace's equation (heat flow). Note, that the observations made in this section will have correspondences in the three-dimensional case.

6. CONCLUSION AND FUTURE WORK

In this paper a Stokes flow based method for measuring cross sections of two-dimensional structures was presented. This method is more versatile than the Laplace equation based approach. A greater variety of domains can be handled; e.g. branching and budding domains. There is no restriction to domains that are enclosed by two simply connected surfaces. A boundary integral approach was used to deal easily with arbitrary shaped boundaries, yielding a clean method, without the need for discretization of the interior of the domain. The method is extendable to three dimensions, where one of the natural geometric measurements would be cross-sectional surface area, and we will expect to find lines of stagnant points (instead of isolated stagnant points at the center of vortices). Future work could include the extension to three dimensional problems, investigations on the handling of stagnant points/lines (the vortices), and center line construction.

7. REFERENCES

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